Cocompact CAT(0) Spaces are Almost Geodesically Complete

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Let M be a Hadamard manifold, that is, a complete simply connected riemannian manifold with non-positive sectional curvatures. Then every geodesic segment $\alpha : [0, a] \rightarrow M$ from $\alpha(0)$ to $\alpha(a)$ can be extended to a geodesic ray $\alpha : [0, \infty) \rightarrow M$. We say then that the Hadamard manifold M is geodesically complete. Note that, in this case, all geodesic rays are proper maps.

CAT(0) spaces are generalizations of Hadamard manifolds. For a CAT(0) space X, all geodesic rays $\alpha : [0, \infty) \rightarrow X$ are proper maps but, in general, X is not geodesically complete. The following definition of almost geodesic completeness was suggested by M. Mihalik:

A geodesic space X, with metric d, is almost geodesically complete if there is a constant C such that for every $p, q \in X$ there is a geodesic ray $\alpha : [0, \infty) \rightarrow X$, $\alpha(0) = p$, and $d(q, \alpha) \leq C$.

In general CAT(0) and hyperbolic spaces are not almost geodesically complete. For instance, the non-negative reals with a line segment of length N attached at the integer N for all $N > 0$ is not almost geodesically complete. In the presence of cocompact group actions the situation changes.

Suppose G is the Cayley graph of an infinite word hyperbolic group. If q is a vertex of G then there is a geodesic line l containing q. For $p \in G$ let r_1 and r_2 be geodesic rays from p to the two limit points of l. This forms an ideal δ-thin triangle and so either r_1 or r_2 must pass within δ of q. Hence G is almost geodesically complete.

This basic fact about word hyperbolic groups is used extensively in the literature (see for example [3]) and Mihalik conjectured an analogous result should be true for CAT(0) groups, that is, groups acting cocompactly by isometries on CAT(0) spaces. For general CAT(0) spaces there are no thin triangles, hence the argument used for word hyperbolic groups above does not work.

In this paper we prove that, under certain conditions, cocompact CAT(0) spaces are almost geodesically complete. In the following statements $H_c^i(X)$ denotes cohomology with integer coefficients and compact supports. Also, a metric space is *proper* if all closed balls are compact.

Our main results are

Theorem A. *Let X be a noncompact proper CAT(0) space on which Γ acts cocompactly*

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by isometries. If $H_c^i(X) \neq 0$, for some i, then X is almost geodesically complete.

Theorem B. *Let X be a noncompact proper CAT(0) space on which Γ acts cocompactly by isometries with discrete orbits. Then X is almost geodesically complete.*

Theorem B follows from theorem A and the following two propositions.

Proposition A. *Let X be a proper CAT(0) space on which Γ acts cocompactly by isometries with discrete orbits. Then X is properly Γ-homotopy equivalent to a Γ-finite Γ-simplicial complex K.*

Proposition B. *Let K be a locally finite contractible simplicial complex which admits a cocompact simplicial action. Then $H_c^i(K) \neq 0$, for some i.*

Proof of Theorem B from Theorem A and Propositions A and B. Let X be a noncompact proper CAT(0) space on which Γ acts cocompactly, by isometries with discrete orbits. By Proposition A, X is properly Γ-homotopy equivalent to a Γ-finite Γ-simplicial complex K. Since every CAT(0) space is contractible, we have that K is also contractible. Hence, by Proposition A, $H_c^i(X) = H_c^i(K) \neq 0$. We can now apply Theorem A and conclude that X is almost geodesically complete.

It was suggested by R. Geoghegan that proposition A above could be used to prove that that the boundary ∂X of a Γ-cocompact CAT(0) space X is a shape invariant of the Γ action. The next theorem shows that in fact this is true.

Theorem C. *Let X and Y be proper CAT(0) spaces on which Γ acts cocompactly by isometries with discrete orbits. If X and Y are Γ-homotopy equivalent then ∂X and ∂Y are shape equivalent.*

Corollary A. *Let X and Y be proper CAT(0) spaces on which Γ acts cocompactly by isometries with discrete orbits. If the actions have the same isotropy, i.e. if $\{ G < \Gamma : X^G \neq \emptyset \} = \{ G < \Gamma : Y^G \neq \emptyset \}$ then ∂X and ∂Y are shape equivalent.*

We say that a group acts on a space with finite isotropy if all isotropy groups are finite.

Corollary B. *Let X and Y be proper CAT(0) spaces on which Γ acts cocompactly by isometries, with discrete orbits and finite isotropy. Then ∂X and ∂Y are shape equivalent.*

It is known that if we assume Γ in theorem C to be hyperbolic, then in fact ∂X and ∂Y are homeomorphic (see [12]), but in general ∂X and ∂Y do not have to be homeomorphic (see [8]).

Here is a short outline of the paper. In section 1 we recall some definitions and a lemma.

In section 2 we prove proposition A, theorem C and its corollaries. In section 3 we prove theorem A. Finally, in section 4 we prove proposition B.

We are grateful to Mike Mihalik for suggesting the problem and for his helpful comments, and to Dan Farley for pointing out some improvements in the exposition. The idea of using nerves in the proof of proposition A was suggested by Stratos Prassidis and also by the work of Bieri and Geoghegan [4]. We are grateful to them. We thank the referee for its useful remarks.

1. Definitions.

For the definitions and basic facts about geodesics and CAT(0) spaces see, for instance, [1] or [5]. Throughout this paper all CAT(0) spaces are assumed to be *complete* metric spaces. Note that this assumption does not affect the statements of our main results since we always take our CAT(0) spaces to be proper. Recall that a metric space is *proper* if all balls are compact.

We say that a group Γ *acts cocompactly* on a space X if there is a compact subset C of X such that $X = \bigcup_{\gamma \in \Gamma} \gamma C$.

If Γ acts on a space X define the *isotropy groups*

$$\Gamma^x = \{\gamma \in \Gamma \ : \ \gamma x = x\}$$

and if $A \subset \Gamma$, define the subgroups

$$\Gamma^A = \{\gamma \in \Gamma \ : \ \gamma \ fixes \ A \ pointwise\}$$

and

$$\Gamma^{(A)} = \{\gamma \in \Gamma \ : \ \gamma A = A\}$$

Define also, for $G \subset \Gamma$, the *fixed point set*

$$X^G = \{x \in X \ : \ gx = x, \ g \in G\}$$

Recall that if X is a CAT(0) space and Γ acts by isometries on X, then X^G is convex, hence contractible. Also, note that $X^G \neq \phi$ iff $G \subset \Gamma^x$, for some $x \in X$.

Let $\mathcal{C}$ be a collection of subgroups of Γ. We say that X is $\mathcal{C}$-*free* if $\Gamma^x \in \mathcal{C}$, for all $x \in X$. Also, we say that X is $\mathcal{C}$-*contractible* if X^G is non-empty and contractible, for $G \in \mathcal{C}$. If Γ acts cellularly on a CW-complex K, we say that K is a *universal* $(\Gamma, \mathcal{C})$-*complex* if K is $\mathcal{C}$-free, $\mathcal{C}$-contractible and $\Gamma^{(\sigma)} = \Gamma^\sigma$, for all cells σ of K.

Lemma 1.1. *Every CAT(0) space is an AR.*

The proof follows from [14], IV.4.1. by taking a refinement of the covering α consisting of convex subsets (for instance balls). Theorem IV.1.2 of [14] also works.

2. Proof of Proposition A, Theorem C and the Corollaries.

2.1. Proposition A. *Let X be a proper CAT(0) space on which Γ acts cocompactly by isometries with discrete orbits. Then X is Γ-homotopy equivalent to a Γ-finite Γ-simplicial complex K.*

Proof. Since the orbits Γx are discrete, we have that for every $x \in X$ there is a closed ball $B_x(r_x)$, with center x and radius $r_x > 0$, such that $B_x(r_x) \cap (\Gamma x) = \{x\}$. Note that this implies that for every $\gamma \in \Gamma$, either $B_x(\frac{r_x}{2}) \cap \gamma B_x(\frac{r_x}{2}) = \emptyset$ or $\gamma x = x$, and this last case implies $B_x(\frac{r_x}{2}) = \gamma B_x(\frac{r_x}{2})$. Also, because X is proper, these balls are compact.

Now, the action is cocompact, thus there is a finite collection $\mathcal{V}$ of balls $B_x(\frac{r_x}{4})$, such that $\bigcup_{\gamma \in \Gamma, V \in \mathcal{V}} \gamma V = X$. Define $\mathcal{U} = \Gamma \mathcal{V} = \{\gamma V : \gamma \in \Gamma, V \in \mathcal{V}\}$.

We show that every ball $U \in \mathcal{U}$ intersects only a finite number of elements in $\mathcal{U}$. Suppose not, then there are $V_1 = B_x(\frac{r_x}{4})$, $V_2 = B_y(\frac{r_y}{4})$ and a sequence $\{\gamma_i\} \subset \Gamma$, such that $\gamma_i V_1$ intersects V_2, and the $\gamma_i V_1$'s are all different. We may assume that $r_x \geq r_y$. Thus, if $\gamma_i B_x(\frac{r_x}{4})$ and $\gamma_j B_x(\frac{r_x}{4})$ both intersect V_2, then $\{y\} \in \gamma_i B_x(\frac{r_x}{2}) \cap \gamma_j B_x(\frac{r_x}{2})$. Therefore $\gamma_i B_x(\frac{r_x}{2})$ intersects $\gamma_j B_x(\frac{r_x}{2})$, for all i, j. Hence $(\gamma_j)^{-1} \gamma_i B_x(\frac{r_x}{2})$ intersects $B_x(\frac{r_x}{2})$ and we mention before that this implies $(\gamma_j)^{-1} \gamma_i x = x$. Follows that the $\gamma_i V_1$'s are all equal, a contradiction. This proves that every ball $U \in \mathcal{U}$ intersects only a finite number of elements in $\mathcal{U}$.

Denote by K the nerve of the covering $\mathcal{U}$. Recall that K is the complex that has one vertex for each element of $\mathcal{U}$, and $\{U_0, ..., U_k\}$ forms a simplex $< U_0, ..., U_k >$ if $U_0 \cap ... \cap U_k$ is non-empty. Note that, since every element in $\mathcal{U}$ intersects only a finite number of elements in $\mathcal{U}$, K is locally finite. Remark that Γ acts on K, simplicially by $\gamma < U_0, ..., U_k > = < \gamma U_0, ..., \gamma U_k >$. It follows that K, with this action, is Γ-finite and cocompact; hence K is finite dimensional.

Claim 1. $\Gamma^{(\sigma)} = \Gamma^\sigma$, *where σ is a simplex of K.*

Let $\{U_0, ..., U_k\}$ be the vertices of σ in K. For $\gamma \in \Gamma^{(\sigma)} = \{\gamma \in \Gamma : \gamma \sigma = \sigma\}$ we have that $\gamma(U_0 \cap ... \cap U_k) = U_0 \cap ... \cap U_k \neq \emptyset$. Hence γU_i intersects U_i, which means, as before, that $\gamma U_i = U_i$. Thus γ fixes σ pointwise and $\Gamma^{(\sigma)} = \Gamma^\sigma$. This proves claim 1.

Claim 2. K^G *is contractible, for $G \subset \Gamma$, and $K^G \neq \emptyset$ iff $X^G \neq \emptyset$.*

Let $G \subset \Gamma$ and let L be the subcomplex of K defined by

$L = \{\sigma \in K : G\sigma = \sigma\} = \{\sigma \in K : G \text{ fixes } \sigma \text{ pointwise}\}$ (by claim 1).

If $x \in K^G$ and $x \in int(\sigma)$, then $G\sigma = \sigma$ (the action is simplicial). Thus $K^G = L$.

Let $\mathcal{W} = \{U \cap X^G : U \in \mathcal{U}, U \cap X^G \neq \emptyset\}$. Recall that X^G is convex. Hence $\mathcal{W}$ is a cover of X^G by convex compact subsets of X^G. Denote by J the nerve of $\mathcal{W}$.

subclaim. L *is homotopy equivalent to J.*

Let U be a vertex of L. Then $\gamma U = U$, for $\gamma \in G$. This means that $\gamma x = x$, for $\gamma \in G$,

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where x is the center of the ball U. Thus $U \cap X^G \neq \emptyset$. Conversely, if $U \cap X^G \neq \emptyset$, then $\gamma U \cap U \neq \emptyset$, hence $\gamma U = U$, for $\gamma \in G$. This defines a surjection $U \mapsto U \cap X^G$, from the vertices of L to the vertices of J.

Now, if $\{U_0, ...U_k\}$ forms a simplex in L (that is, if $\cap U_i \neq \emptyset$) then $G[\cap U_i] = \cap U_i$. But then $Gp = p$, where $p \in \cap U_i$ is the center of the compact convex set $\cap U_i$ (the center is unique, see [2], p.10). Thus $\cap(U_i \cap X^G) = \cap U_i \cap X^G \neq \emptyset$.

Hence, the surjection $U \mapsto U \cap X^G$, from the vertices of L to the vertices of J defines a simplicial map $h : L \to J$. We show that $h^{-1}(\sigma)$ is a simplex, for a simplex σ of J. So, let $\sigma = < U_1 \cap X^G, ..., U_k \cap X^G > \in J$. Then the vertices of $h^{-1}(\sigma)$ are $U'_1, ..., U'_l \in \mathcal{U}$, where, for every $j = 1, ..., l$, there is some $i = 1, ..., k$ with $U'_j \cap X^G = U_i \cap X^G$. But then $\emptyset \neq (\cap U_i) \cap X^G = \cap U'_j \cap X^G \subset \cap U'_j$, which means that $h^{-1}(\sigma)$ is a simplex.

Consequently, h is a proper cellular map. Hence h is a homotopy equivalence. This proves the subclaim.

Note that the proof of the subclaim implies $K^G \neq \emptyset$ iff $X^G \neq \emptyset$. To finish the proof of the claim note that $\mathcal{W}$ is a brick decomposition of X^G in the sense of [9]: nonempty intersections of elements in $\mathcal{W}$ are compact and convex. Hence, by lemma 1.1, they are AR's. Then the main result in [9] implies that J is homotopy equivalent to X^G which is contractible. This completes the proof of the claim.

Let $\mathcal{C}$ be the collection of subgroups of Γ given by

$$\mathcal{C} = \{G \; : \; X^G \neq \emptyset\} = \{G \; : \; G \text{ subgroup of } \Gamma^x, \text{ for some } x \in X\}$$

The second statement of claim 2 implies

$$\mathcal{C} = \{G \; : \; K^G \neq \emptyset\} = \{G \; : \; G \text{ subgroup of } \Gamma^a, \text{ for some } a \in K\}$$

This, together with the first statement of claim 2, imply that K is a universal $(\Gamma, \mathcal{C})$-complex. Also note that, since X^G is contractible and non-empty, for every $G \in \mathcal{C}$, X is $\mathcal{C}$-free and $\mathcal{C}$-contractible. Hence there is a Γ-map $f : K \to X$ (see [11], p.286, and note that there it is not required for X to be a complex).

Recall also the definition of the canonical g map of a space into its nerve: for $x \in X$, let $U_0, ..., U_k \in \mathcal{U}$, be the set of all elements in $\mathcal{U}$ that contain x. Then the barycentric coordinates $\lambda_0, ..., \lambda_k$ of $g(x)$ in the simplex $< U_0, ..., U_k >$ are

$$\lambda_i = \frac{d(x, X \setminus U_i)}{\sum_{j=0}^{j=k} d(x, X \setminus U_j)}$$

where d denotes the metric on X. Note that g is a Γ-map: if $U_0, ..., U_k$, is the set of all elements in $\mathcal{U}$ that contain x, then $\gamma U_0, ..., \gamma U_k$, is the set of all elements in $\mathcal{U}$ that contain γx, and

$$\frac{d(\gamma x, X \setminus \gamma U_i)}{\sum_{j=0}^{j=k} d(x, X \setminus \gamma U_j)} = \frac{d(x, X \setminus U_i)}{\sum_{j=0}^{j=k} d(x, X \setminus U_j)}$$

This means that $gf : K \to K$ is a Γ-map, and from K being a universal $(\Gamma, \mathcal{C})$-complex we deduce that gf is Γ-homotopic to the identity on K (see [11], p.286, theorem A.2.)

We want to prove now that fg is Γ-homotopic to the identity in X, but we can not apply the argument above because we do not have that X is a Γ-complex. But now we use the fact that X is a CAT(0) space.

Lemma 2.2. *Let Z be a Γ-space and X be a CAT(0) space on which Γ acts by isometries. Then any two Γ-maps from Z to X are Γ-homotopic.*

Proof. Let $f_0, f_1 : Z \to X$, be two Γ-maps. Define $f_t(z) = \alpha_z(d(f_0(z), f_1(z)) \, t)$, for $z \in Z$ and $t \in [0, 1]$, where α_z is the unique geodesic beginning at $f_0(z)$ and ending at $f_1(z)$. It follows from well known facts about geodesics on CAT(0) spaces (see for instance [1], [5]) that f is a continuous Γ-map.

Since X is proper, it is straightforward to show that all maps and homotopies are proper. This completes the proof of the proposition.

Remark. The condition of X being proper is necessary. For take X to be the cone over the integers and extend the action of the integers $\mathbb{Z}$ on itself to the CAT(0) space X with fixed point the vertex. Then $\mathbb{Z}$ acts cocompactly by isometries and with discrete orbits on X but X is not *properly* Γ-homotopic to a Γ-finite Γ-simplicial complex. Note that X is Γ-homotopic to a point, but not properly Γ-homotopic to a point.

Now, before applying the proposition to prove theorem C we need some comments and a lemma. Let X be a proper CAT(0) space on which Γ acts cocompactly by isometries and with discrete orbits and let K be a Γ-finite Γ-simplicial complex Γ-homotopy equivalent to X. Let K_n, $n = 1, 2, 3...$ be a sequence of subcomplexes of K such that they satisfy:

(i) $K_{n+1} \subset K_n$
(ii) $\overline{K \setminus K_n}$ *is compact*
(iii) $\bigcap K_n = \phi$

Let $\iota : K_{n+1} \to K_n$ be the inclusion and denote by $\mathcal{K}$ the tower $\{K_1 \xleftarrow{\iota} K_2 \xleftarrow{\iota} K_3 \xleftarrow{\iota} ...\}$ (i.e. a inverse system indexed by $\mathbb{N}$, see [10]). We say that $\mathcal{K}$ is a (X, K)-tower.

Lemma 2.3. *Every (X, K)-tower is a polyhedral resolution of ∂X.*

Remark. For the definition of a resolution see [15]. This lemma implies that the shape of ∂X is determined by $\mathcal{K}$ in *pro-H*, where H is the homotopy category.

Proof of the lemma. Let d be the metric on X and fix a point $x_0 \in X$. Write $B_n = B(x_0, n) = \{x \in X : d(x, x_0) \leq n\}$, $S_n = \{x \in X : d(x, x_0) = n\}$ and $X_n = \{x \in X : d(x, x_0) \geq n\}$, for $n = 1, 2, 3...$ Let $\mathcal{S}$ be the tower $\{S_1 \xleftarrow{r} S_2 \xleftarrow{r} S_3 \xleftarrow{r} ...\}$, where the maps $r : S_{n+1} \to S_n$ are given by geodesic retraction. Because $\partial X = \lim_{\leftarrow} S_n$ and all S_n are compact theorem 5 of [15] implies that the tower $\mathcal{S}$ is a resolution of ∂X. But geodesic retraction also induces an equivalence, in *pro-H*, between $\mathcal{S}$ and $\mathcal{X} = \{X_1 \xleftarrow{j} X_2 \xleftarrow{j} X_3 \xleftarrow{j} ...\}$, where

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the j's are the inclusions. Hence $\mathcal{X}$ is also a resolution of ∂X. It remains to prove that $\mathcal{X}$ and $\mathcal{K}$ are equivalent in *pro-H*. Let $f : X \to K$ and $g : K \to X$ be Γ-maps such that fg and gf are Γ-homotopic to the corresponding identities. By taking subsequences we can assume that the following conditions hold.

(1) $f(X_n) \subset K_n$

(2) $g(K_{n+1}) \subset X_n$

(3) $gf \mid_{X_{n+1}} : X_{n+1} \to X_n$ and $fg \mid_{K_{n+1}} : K_{n+1} \to K_n$ are Γ-homotopic to the inclusions $X_{n+1} \to X_n$ and $K_{n+1} \to K_n$, respectively.

Hence, by (1) and (2) above we can define maps $F : \mathcal{X} \to \mathcal{K}$ and $G : \mathcal{K} \to \mathcal{X}$ given by $F_n = f \mid_{X_n} : X_n \to K_n$ and $G_n = g \mid_{K_n} : K_n \to X_{n-1}$. Also by (3) above we get that $GF : \mathcal{X} \to \mathcal{X}$ is equivalent in *pro-H* to the inclusion map $\mathcal{X} \to \mathcal{X}$ (i.e. given by the inclusions $j : X_{n+1} \to X_n$). But $\mathcal{X} = \{X_1 \xleftarrow{j} X_2 \xleftarrow{j} X_3 \xleftarrow{j} ...\}$, consequently the inclusion map $\mathcal{X} \to \mathcal{X}$ is equivalent in *pro-H* to the identity on $\mathcal{X}$. In the same way we also get that FG is equivalent to the identity on $\mathcal{K}$. This proves the lemma.

Proof of theorem C. By proposition A, X admits a (X, K)-tower $\mathcal{K}$. But X is Γ-homotopy equivalent to Y, hence Y is also Γ-homotopy equivalent to K. All this together with lemma 2.3 imply that $\mathcal{K}$ is a polyhedral resolution of both, ∂X and ∂Y. This proves theorem C.

For a group Γ acting on a space Z, let $\mathcal{C}_Z = \{G < \Gamma \ : \ Z^G \neq \emptyset\}$. Corollary A follows from the following lemma.

Lemma 2.4. *Let X and Y be proper CAT(0) spaces on which Γ acts cocompactly by isometries and with discrete orbits. Then the actions have the same isotropy (i.e. $\mathcal{C}_X = \mathcal{C}_Y$) if and only if X is Γ-homotopy equivalent to Y.*

Proof. Suppose that X is Γ-homotopy equivalent to Y. Then there is a Γ-map $f : X \to Y$. Let $g \in \Gamma$ and $x \in X$ such that $gx = x$. Then $gy = y$, for $y = f(x)$. This implies $\mathcal{C}_X \subset \mathcal{C}_Y$. In the same way we obtain $\mathcal{C}_Y \subset \mathcal{C}_X$. Hence both actions have the same isotropy.

Conversely, suppose $\mathcal{C}_X = \mathcal{C}_Y = \mathcal{C}$. Then, by proposition A, X and Y are properly Γ-homotopy equivalent to Γ-finite Γ-simplicial complexes K and J, respectively. In fact, in the proof of proposition A we showed that $\mathcal{C}_X = \mathcal{C}_K$ and $\mathcal{C}_Y = \mathcal{C}_J$. Hence $\mathcal{C}_K = \mathcal{C}_J = \mathcal{C}$. But in the proof of proposition A we also showed that K and J are universal $(\Gamma, \mathcal{C})$-complexes, which are unique, up to Γ-homotopy equivalence (see [11], p.286). This proves the lemma and corollary A.

Proof of Corollary B. First recall that if G is a finite group acting on a proper CAT(0) space X, then G fixes some point p. This point is the unique center of the compact G-invariant set Gp (see [2], p.10). This implies $G \in \mathcal{C}_X$, for every G finite. Hence, if the isotropy $\mathcal{C}_X$ consists only of finite subgroups, then it is the family of *all* finite subgroups. Consequently, if Γ acts on X and Y with finite isotropy (i.e. all isotropy groups are finite),

then $\mathcal{C}_X = \mathcal{C}_Y = \{G < \Gamma : G$ is finite $\}$. The corollary now follows from corollary A.

3. Proof of theorem A.

Theorem A. *Let X be a noncompact proper CAT(0) space on which Γ acts cocompactly by isometries. If $H_c^i(X) \neq 0$, for some i, then X is almost geodesically complete.*

Proof. Here is the idea of the proof. If X is not almost geodesically complete we construct (see claim 2 below) retractions f_r, $r > 0$, properly homotopic to the identity, such that $f_r(X)$ misses an arbitrarily large ball about a fixed point p (the balls get larger as $r \to \infty$). Then we prove in claim 3 that $H_c^*(X) = 0$, in the following way. First we prove that for $z \in H_c^*(X)$ there is an $r > 0$ such that $f_r^*(c) = 0$ (for this just take r large enough so that $f_r(X)$ misses the support of z). But f_r is properly homotopic to the identity 1_X, hence $z = 1_X^* z = f_r^* z = 0$. We now give a detailed proof.

Given $p \in X$ and $s > 0$ define $f^{p,s} : X \to X$, by

$$f^{p,s}(\alpha(t)) = \begin{cases} p = \alpha(0) & t \leq s \\ \alpha(t - s) & t \geq s \end{cases}$$

where α is a geodesic beginning at p. (This makes sense because for every two points in a CAT(0) space there is a unique geodesic segment joining these two points and depending continuously on them. See [1], [5].) Note that $f^{p,s}$ is a proper map properly homotopic to the identity in X.

Also, given a geodesic segment α denote by ℓ_α the supremum over all ℓ such that α can be extended to the interval $[0, \ell]$. If the maximum does not exist we write $\ell_\alpha = \infty$. Note that if $\ell_\alpha < \infty$, there is a geodesic segment defined on the interval $[0, \ell_\alpha]$ extending α (because X is proper), and if $\ell_\alpha = \infty$ then α can be extended to $[0, \infty)$. (For this last statement we can use a general Arzelá Ascoli argument, or, for simplicity in this special case, the fact that $X \cup \partial X$ is compact. For the definition of the boundary ∂X and its properties, see for instance, [5]).

Fix $p \in X$. Now, since the action is cocompact, to prove the theorem it is enough to prove the following:

There is a constant C such that for every $\gamma \in \Gamma$ there is a geodesic ray $\alpha : [0, \infty) \to X$, $\alpha(0) = p$, and $d(\gamma(p), \alpha) \leq C$.

We prove this by contradiction. Assume that the statement above does not hold for X. We have

(*) Given $r > 0$ there is $\gamma_r \in \Gamma$ such that $\ell_{[p,x]} < \infty$ for $x \in B_{\gamma_r(p)}(r)$.

Here $B_{\gamma_r(p)}(r)$ denotes the closed ball with center $\gamma_r(p)$ and radius r and $[p, x]$ denotes the (unique) geodesic segment from p to x.

Claim 1. Given $r > 0$, we have

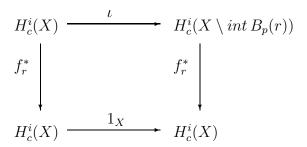$$sup\{\ell_{[p,x]} \ : \ x \in B_{\gamma_r(p)}(r)\} \ < \ \infty$$

We prove this by contradiction. Suppose there is a sequence $\{x_j\} \subset B_{\gamma_r(p)}(r)$ with $\ell_{[p,x_j]} \to \infty$ and consider the sequence $\{y_j\}$, defined by $y_j = \alpha(\ell_{[p,x_j]})$, where α is a geodesic segment extending $[p, x_j]$ (choose one α). Because $X \cup \partial X$ is compact we can assume that the sequence $\{y_j\}$ converges to a point x_0 in the boundary ∂X of X. Then the geodesic ray $[p, x_0)$ intersects $B_{\gamma_r(p)}(r)$, which contradicts (*).

Claim 2. Given $r > 0$ there is $f_r : X \to X$, properly homotopic to the identity, such that $f_r(X) \subset X \setminus B_p(r)$

Note that, since X is noncompact (and claim 1), we can assume that $p \notin B_{\gamma_r(p)}(r)$. Write $s = sup\{\ell_{[p,x]} \ : \ x \in B_{\gamma_r(p)}(r)\} \ < \ \infty$, and remark that $f^{p,s}(X) \subset X \setminus B_{\gamma_r(p)}(r)$. Hence $\gamma_r^{-1} f^{p,s}(X) \subset X \setminus B_p(r)$. Take $f_r = \gamma_r^{-1} f^{p,s}$. This proves the claim.

Claim 3. $H_c^i(X) = 0$, for all i.

Take a $z \in H_c^i(X)$. Then there is a $r > 0$ such that $\iota(z) = 0$, where $\iota : H_c^i(X) \to H_c^i(X \setminus int\, B_p(r))$ is the restriction map. Take f_r from claim 2 and consider the following commutative diagram corresponding to the map of pairs $f_r : (X, X) \to (X, X \setminus int\, B_p(r))$,

$$
\begin{array}{ccc}
H_c^i(X) & \xrightarrow{\ \iota\ } & H_c^i(X \setminus int\, B_p(r)) \\
\Big\downarrow{\scriptstyle f_r^*} & & \Big\downarrow{\scriptstyle f_r^*} \\
H_c^i(X) & \xrightarrow{\ 1_X\ } & H_c^i(X)
\end{array}
$$

where 1_X denotes the identity. Then $f_r^*(z) = f_r^*(\iota(z)) = 0$. But f_r is properly homotopic to the identity, thus f_r^* is the identity and $z = 0$. This completes the proof of Theorem A.

4. Proof of Proposition B.

Before proving theorem B, we recall some definitions and results from PL topology (see [16]).

Let X be a PL space. A triangulation on X is a pair (T, ρ), where T is a simplicial complex and $\rho : T \to X$ is a PL homeomorphism. Sometimes we just say that T is a triangulation of X. For a simplicial complex T, $\mid T \mid$ denotes the underlying topological space. From now on, all simplicial complexes are locally finite and finite dimensional.

Let $K \subset T$ be simplicial complexes. We say that K is full in T if every simplex of T intersects K in exactly one (possibly empty) face. Define $N(T, K)$ to be the subcomplex of T that contains all simplices intersecting $\mid K \mid$, together with their faces and $C(T, K)$ the subcomplex of T that contains all simplices not intersecting $\mid K \mid$. Define also $int\, N(T, K) = \mid N(T, K) \mid \setminus \mid C(T, K) \mid$.

Let Y be a PL space and $X \subset Y$ a PL subspace. Then $N = \mid N(T', K) \mid$ is called a regular neighborhood of $\mid K \mid$ in $\mid T \mid$, where: T is a triangulation of Y and K the subcomplex of T triangulating X with K full in T, and T' is a first derived of T near K (i.e T' is obtained from T by subdividing only simplices that are neither in K, nor in $C(T, K)$, see p.32 of [16]).

Let Δ^n and Δ^m be standard simplices. Then $\Delta^n * \Delta^m = \Delta^{n+m+1}$, where the star denotes "join" (see [16], p.2). Note that every $z \in \Delta^{n+m+1}$ can be written uniquely as $z = (1-s)x + sy$, $s \in [0,1]$, $x \in \Delta^n$, $y \in \Delta^m$ (here we are assuming $\Delta^n, \Delta^m \subset \mathbb{R}^N$ in general position). Note also that for $s = 0$ we get the points of Δ^n, and for $s = 1$ we get the points of Δ^m.

Define, for $t \in [0,1]$, the canonical deformation retraction of $c_t : (\Delta^{n+m+1} \setminus \Delta^m) \to (\Delta^{n+m+1} \setminus \Delta^m)$ by $c_t((1-s)x + sy) = ((1-ts)x + tsy)$, $x \in \Delta^n$, $y \in \Delta^m$, $s \in [0, 1)$ (note that c_t is well defined and continuous for $s \neq 1$). Hence c_1 is the identity, $c_0(\Delta^{n+m+1} \setminus \Delta^m) \subset \Delta^n$ and $c_t(x) = x$ for all $t \in [0,1]$ and $x \in \Delta^n$.

Remarks.
1. Let d be any metric on the simplex Δ^{n+m+1}, compatible with the topology of Δ^{n+m+1}. We will use the following simple fact: given $\delta > 0$ there is an open neighborhood V_δ of Δ^n in Δ^{n+m+1}, such that $d(c_1(x), c_1(y)) \leq d(x,y) + \delta$, for $x, y \in V_\delta$. (Proof: because $c_1(x) = x$ for $x \in \Delta^n$ there is a neighborhood V_δ of Δ^n such that $d(c_1(x), x) \leq \delta/2$, for $x \in V_\delta$. Then, for $x, y \in V_\delta$ we get $d(c_1(x), c_1(y)) \leq d(c_1(x), x) + d(x,y) + d(y, c_1(y)) \leq \delta + d(x,y)$.)

2. Let K be full in T. Then every simplex Δ in $N(T, K) \setminus (C(T, K) \cup K)$ can be written uniquely as $\sigma * \tau$, where $\sigma < \Delta$ is the simplex $\Delta \cap K$ and τ is the *complementary* simplex $\Delta \cap C(T, K)$. Hence using c_t defined above, we can construct simplexwise a deformation retraction $c_t = c_t(T, K) : int(N(T, K)) \to int(N(T, K))$ with the same properties, that is, c_1 is the identity, $c_0(int(N(T, K))) \subset \mid K \mid$ and $c_t(x) = x$ for all $t \in [0,1]$ and $x \in |K|$. Write $c = c_0$.

Let T be a triangulation of the PL space Y. For every simplex $\sigma \in T$ choose a flat metric d_σ on it (i.e. there is a linear isometric embedding from (σ, d_σ) into some euclidean space) in such a way that the metrics coincide on each intersection of simplices. This gives a way to define the length of a path in Y and this determines a metric d on Y by defining $d(x, y) = inf\{$lengths of paths joining x to y $\}$. We say that d is a piecewise flat metric on the PL space Y (or a PL metric on Y), with respect to the triangulation T. We have that if T is locally finite and (Y, d) is complete (as a metric space) then Y with metric d is a geodesic space. (Recall that we are assuming all simplicial complexes to be locally finite.)

Let Y be a PL space with a PL metric d, where d is piecewise flat with respect to the triangulation T of Y, and let L be a subcomplex of T. We denote by d_L the intrinsic metric on $|L|$ induced by d, i.e. $d_L(x,y) = inf\{$lengths of paths in L joining x to y $\}$. Let U be any other triangulation of Y and J a subcomplex of U, with $|J| \subset |L|$. Define $mesh_L(J) = sup\{diam_L(\sigma) : \sigma \in J\}$, where $diam_L(\sigma)$ is the diameter of $|\sigma| \subset |L|$ with respect to the metric d_L. If $L = T$ we simply write $mesh$ instead of $mesh_L$. Also, define $mesh_0(J) = sup\{length(\sigma) : \sigma \in J^1\}$, where J^1 is the set of one simplices in J. Note that if $J, L, |J| \subset |L|$, are subcomplexes of a subdivision of T, then $mesh_L(J) \le mesh_0(J) \le mesh_0(T)$.

4.1. Lemma. *Let N_1 and N_2 be regular neighborhoods of the PL space X in the PL space Y, with piecewise flat metric d. Then there is a PL homeomorphism $j : Y \to Y$, with $j(N_1) = N_2$, which is the identity on X. Moreover if $N_i = |N(T_i', K_i)|$, where K_i is induced by T_i, $i = 1, 2$, we have*

$$d(x, j(x)) \le mesh_Y(T_1') + mesh_Y(T_2')$$

(Here T_i' is a first derived of T_i near K_i, $i = 1, 2$.)

Proof. The first part is theorem 3.8 of [16] (drop compactness). The inequality follows because the map constructed in the proof of theorem 3.8 of [16] is a composition of two maps, each moving a point in $\sigma \cap N_i$, where σ is a simplex of $N(T_i, K_i)$, $i = 1, 2$. This proves lemma 4.1.

Denote the map j given by the lemma as $j(T_1', T_2')$.

We will also need the following result (see [16], p.32)

4.2. Lemma. *Let X be a PL subspace of the PL space Y, T a triangulation of Y and K a subcomplex of T triangulating X, where K is full in T. Given an open neighborhood U of X in Y there is a first derived T' of T near K, such that the regular neighborhood $|N(T', K)|$ is contained in U.*

4.3. Proposition B. *Let K be a locally finite, contractible simplicial complex which admits a cocompact simplicial action. Then $H_c^i(|K|) \neq 0$, for some i.*

Proof. First we give some motivation and the idea of the proof.

We will prove the proposition by contradiction, so we will assume $H_c^i(|K|) = 0$, for all i. Now, consider first a particular case: suppose that K is PL homeomorphic to a PL manifold M^n. Since M^n is contractible and $H_c^i(M) = 0$, for all i, we can assume, after some stabilization, that M^n is PL-homeomorphic to euclidean half n-space $\mathbb{R}_+^n = \{(x_1, ..., x_n) : x_n \ge 0\}$ and ∂M is PL-homeomorphic to $\mathbb{R}^{n-1}$. Denote by Γ the group that acts simplicially and cocompactly on K and let d be a Γ-invariant metric on $K \cong_{PL} M$. We will prove these two key facts:

(a) M is d-close to ∂M (this is **(2)** in the proof),

(b) bounded extensions of maps to ∂M(for a precise statement see claim 4).

Then we proceed as follows (see arguments after the proof of claim 4). Take $x_0 \in \partial M$. Let D denote a closed PL n-ball. We can construct a PL embedding $\rho : (D, \partial D) \to (M, \partial M)$ such that

(i) $\rho(D)$ is far from x_0

(ii) $\rho|_{\partial D} \neq 0 \in \pi_{n-2}(\partial M \setminus \{x_0\})$.

Using (a), (b) and ρ above we construct a map $\psi : D \to \partial M$ with $\rho|_{\partial D} = \psi|_{\partial D}$ and $\psi(D) \subset \partial M \setminus \{x_0\}$. Hence $\rho|_{\partial D} = \psi|_{\partial D} = 0 \in \pi_{n-2}(\partial M \setminus \{x_0\})$, a contradiction.

For the general case, K may not be PL homeomorphic to a PL manifold. Hence we "replace" K by its regular neighborhood M, in some euclidean space. We do this at the beginning of the proof. The problem now is that Γ does not act, at least a priori, simplicially on M. To prove (b) above (i.e. claim 4 in the proof) we use an "approximate" action of Γ on M (see claim 2 in the proof).

We now give a detailed proof.

The proof is by contradiction. So suppose $H_c^i(|K|) = 0$, for all i, and denote by Γ the group that acts simplicially and cocompactly on K.

Now, note that, since K is locally finite and connected, K is countable, that is, it has a countable number of simplices. Note also that K is finite dimensional. Hence we can embed K simplicially and properly in some $\mathbb{R}^n$, $n \geq 2(\dim K) + 2$, and take $n \geq 6$. Let T be a triangulation of $\mathbb{R}^n$ such that there is a full subcomplex J of T with $|K| = |J|$ and J is a subdivision of K. Give K the unit metric, that is, the geodesic piecewise flat metric where every edge has length one. Note that this metric is Γ-invariant and that K, with this metric, is proper. Denote this piecewise flat metric on $|J| = |K|$ by d_K. By lemma 2.5 of [13] (the proof works for infinite complexes), maybe after a subdivision of T away from K, we can extend this proper piecewise flat metric to a proper piecewise flat metric d on $\mathbb{R}^n = |T|$. Also, maybe after further subdivision, we can assume $mesh_0(T) \leq 1$, and that every simplex of J is convex in $|T|$ (see [5]).

Recall that T' is a first derived of T near J (see [16], p.32). Let $M = |N(T', J)|$ be a regular neighborhood of $|J|$ in $\mathbb{R}^n$. Then M is a n-manifold with boundary ∂M and M is properly homotopic to $|K|$. Hence M is contractible and $H_c^i(M) = H_c^i(|K|) = 0$, for all i. By duality we have that $H_i(\partial M) = 0$, $i > 0$. Note that because M is contractible, the boundary of the regular neighborhood of M in $\mathbb{R}^{n+1}$ is the suspension of the regular neighborhood of M in $\mathbb{R}^n$, where the embedding $M \to \mathbb{R}^{n+1}$ is the composition $M \to \mathbb{R}^n \to \mathbb{R}^{n+1}$. Hence, by embedding canonically $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ if necessary, we can assume that ∂M is simply connected. This implies, together with $H_i(\partial M) = 0$, $i > 0$, that ∂M is contractible.

We claim that we can also assume that M and ∂M are simply connected at infinity. To see this, just cross Γ with $\mathbb{Z}^2$ and K with $\mathbb{R}^2$ if necessary and make $\Gamma \times \mathbb{Z}^2$ act cocompactly on $K \times \mathbb{R}^2$. Note that we still have $H_c^i(K \times \mathbb{R}^2) = 0$, for all i. Note also that $X \times \mathbb{R}^2$ is simply connected at infinity for X simply connected.

Recall that if X^m is a contractible simply-connected-at-infinity high-dimensional PL-manifold with empty boundary, then X is PL-homeomorphic to euclidean m-space. Moreover, if X^{m+1} is a contractible simply-connected-at-infinity high-dimensional PL-manifold

with boundary PL-homeomorphic to euclidean m-space, then X is PL-homeomorphic to half euclidean $(m+1)$-space (see [17], [6]). It follows from these last remarks that we can assume ∂M to be PL-homeomorphic to euclidean $(n-1)$-space $\mathbb{R}^{n-1}$ and M to be PL-homeomorphic to euclidean half n-space $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 \geq 0\}$.

Denote by d_M the intrinsic metric on M induced from $\mathbb{R}^n$ with metric d. Because $|J| \subset M$ we have $d_M|_J \leq d_K$.

Let c be the retract $c = c(T, J) : int(N(T, J)) \to int(N(T, J))$ defined in the introduction of this section. Note that $c = c(T, J)$ depends on the choice of $M = |N(T', J)|$, and the choice of M depends only on the choice of the first derived T'.

Remark that (M, d_M) is a proper metric space. (This is because T is finite dimensional locally finite and proper, hence the same is true for any subcomplex of a (locally finite) subdivision of T.)

We prove now that we can choose M close to $|J|$ to get d_K close to $d_M|_J$. In fact M will get closer and closer to $|J|$, as we approach infinity.

Claim 1. We can choose M so that $d_K \leq (d_M|_J) + 1$

Enumerate all simplices of J: $\sigma_1, \sigma_2, ...,$ and let $\Delta_1, \Delta_2, ...$ their corresponding simplices in $N(T, J)$ of maximal dimension (see remark 2 at the beginning of this section). Let $A = A(n)$ be the number of simplices of the first barycentric subdivision of a n-simplex Δ^n. Remark 1 at the beginning of this section imply that there are open neighborhoods V_i of σ_j in Δ_j, such that for all $x, y \in V_i$ we have

$$d_{\sigma_j}(c(x), c(y)) \leq d_{\Delta_j}(x, y) + \frac{1}{A2^j}$$

Define $W = int \bigcup V_i$. Since all triangulations here are locally finite, we have that W is an open neighborhood of $|J|$. We can assume $\overline{W} \subset int N(T, J)$, so that c is defined at all points of W.

By lemma 4.2 we can choose a first derived T' of T near J such that $M = |N(T', J)| \subset W$. Note that c is defined for every point in $M \subset W$. From the definition of first derived follows that for every $\Delta_j \in N(T, J)$ there is at most A simplices $\Delta' \in N(T', J)$ with $\Delta' \subset \Delta_j$.

Since we took T small enough so that every simplex is convex, we have that $d_K(x, y) = d_{\sigma_j}(x, y)$ if $x, y \in \sigma_j \in J$ and $d_M(x, y) = d_{\Delta'}(x, y)$ if $x, y \in \Delta' \in N(T', J)$. Hence we can write the above inequality in the following form:

$$d_K(c(x), c(y)) \leq d_M(x, y) + \frac{1}{A2^j}$$

for $x, y \in \Delta' \in N(T', J)$ and $\Delta' \subset \Delta_j \in N(T, J)$.

Now, for $x, y \in |J|$, let $\alpha : [0, a] \to M$, $\alpha(0) = x$ and $\alpha(a) = y$, be a distance minimizing path with respect to d_M, and let $0 = t_0 < t_1 < ... < t_r = a$ be such that for each $i = 1, ..., r$ there is a simplex $\Delta'_i \in N(T', K)$, with $\alpha(t_{i-1}), \alpha(t_i) \in \Delta'_i$ and all Δ'_i different. Note that we can choose all Δ'_i different because every simplex of T' is convex. Also, for each i, let j_i be such that $\Delta'_i \subset \Delta_{j_i}$ Write $x_i = \alpha(t_i)$. Thus $d_M(x, y) = \sum d_M(x_{i-1}, x_i)$. For $x, y \in |J|$ we have

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$$d_K(x, y) \leq \sum d_K(c(x_{i-1}), c(x_i)) \leq \sum d_M(x_{i-1}, x_i) + \frac{1}{A2^{j_i}} \leq d_M(x, y) + 1$$

For the first inequality use triangular inequality plus the fact that $c(x_0) = c(x) = x$ and $c(x_r) = c(y) = y$. For the last inequality use the fact that for each j_i there are, at most, A simplices $\Delta_i' \in N(T', J)$ with $\Delta_i' \subset \Delta_{j_i}$.

This proves claim 1.

Note that, because we can make $mesh_0(T)$ small, we can assume

(1) $\qquad d_M(x, c(x)) \leq 1$, and

(2) $\qquad$ for every $x \in M$ there is $w \in \partial M$ with $d_M(x, w) \leq 1$ (take $w = c(x)$ and use **(1)**).

Recall that Γ acts on (K, d_K) and on (J, d_K) simplicially by isometries.

Claim 2. For every $\gamma \in \Gamma$, there is a PL homeomorphism $g : M \to M$, such that $d_K(cg(x), \gamma c(x)) \leq 15$, for all $x \in M$.

Let $\iota : |J| \to \mathbb{R}^n$ denote the inclusion. Because we assumed $n \geq 2(dim\, J) + 2$ the two embeddings ι and $\iota\gamma$ are ambient isotopic. Then, by the uniqueness of regular neighborhoods, there is a PL-homeomorphism $g_1 : M \to M$, such that $g_1|_K = \gamma$.

Now, because $cg_1c_t(x) \to cg_1c(x) = \gamma c(x)$, as $t \to 1$, for every $x \in M$, there is an open neighborhood $U \subset M$ of $|J|$ such that,

(3) $\qquad\qquad d_K(cg_1(x), \gamma c(x)) < 1 \quad x \in U$

By lemma 4.2 there is a first derived T'' of T near J with

(4) $\qquad\qquad N = |N(T'', J)| \subset U$.

Note that, because $mesh_0(T) \leq 1$, we have that $mesh_M(T') \leq 1$ and $mesh_M(T'') \leq 1$ (see comment right before lemma 4.1). Now, let $h = j(T', T'')$, given by lemma 4.1, and we get $d_M(x, h(x)) \leq 2$, for $x \in M$. This, together with **(1)** and claim 1, imply, for $x \in M$,

(5) $\qquad d_K(ch(x), c(x)) \leq d_M(ch(x), c(x)) + 1$
$\qquad\qquad\qquad\qquad \leq d_M(ch(x), h(x)) + d_M(h(x), x) + d_M(x, c(x)) + 1 \leq 5$

Let also $j = j(g_1T'', T')$, so $j(g_1N) = M$. If σ is a simplex in $N(T'', J)$, **(1)**,**(3)**, **(4)** imply:

$$diam_M(g_1\sigma) \leq diam_M(cg_1\sigma) + 2$$
$$\leq diam_K(cg_1\sigma) + 2 \leq diam_K(\gamma c\sigma) + 4$$
$$\leq diam_K(c\sigma) + 4 \leq mesh_0(T) + 4 \leq 5.$$

Here $diam_K(A)$ is the diameter of $A \subset |J| = |K|$, with metric d_K.

Hence, by lemma 4.1 we get

$$d_M(x, j(x)) \leq mesh_M(g_1 T'') + mesh_M(T') \leq 5 + 1 = 6$$

This together with **(1)** and claim 1 imply for $x \in g_1 N$,

$$(\textbf{6}) \qquad d_K(c(x), cj(x)) \leq \big[d_M(c(x), x) + d_M(x, j(x)) + d_M(j(x), cj(x)) \big] + 1$$

$$\leq (1 + 6 + 1) + 1 = 9$$

Define $g = jg_1 h$. Then **(3)**,**(4)**,**(5)**, and **(6)** imply, for $x \in M$,

$$d_K(cg(x), \gamma c(x)) = d_K(cjg_1 h(x), \gamma c(x))$$

$$\leq d_K(cj[g_1 h(x)], c[g_1 h(x)]) + d_K(cg_1 h(x), \gamma c(x))$$

$$\leq 9 + d_K(cg_1[h(x)], \gamma c[h(x)]) + d_K(\gamma ch(x), \gamma c(x))$$

$$\leq 9 + 1 + d_K(\gamma ch(x), \gamma c(x)) = 10 + d_K(ch(x), c(x)) \leq 10 + 5 = 15.$$

This proves claim 2.

Claim 3. Given $b > 0$, there is an a_b, such that the following holds. For any map $\phi : \partial\sigma \to \partial M$, where σ is a simplex, with $diam_M(\phi(\partial\sigma)) \leq b$, there is an extension $\phi : \sigma \to \partial M$ with $diam_M(\phi(\sigma)) \leq a_b$.

Let C_0 be a finite subcomplex of J such that for any subset C' of J, with $diam_K(C') \leq b + 3$, there is a $\gamma \in \Gamma$ such that $C' \subset \gamma C_0$. Let also C_1 be a finite subcomplex of J such that $d_K(J \setminus C_1, C_0) \geq 16$.

Consider now $c^{-1}(C_1) \cap \partial M$. There is a $B \subset \partial M$, homeomorphic to a $(n-1)$ ball such that $\partial M \cap c^{-1}(C_1) \subset B$ (recall that ∂M is homeomorphic to $\mathbb{R}^{n-1}$ and c is proper). Let E_0 be a finite subcomplex of K such that $B \subset c^{-1}(E_0)$. Then for any map $\partial\sigma \to c^{-1}(C_1) \cap \partial M \subset B$ there is an extension $\sigma \to c^{-1}(E_0) \cap \partial M$. Let also E_1 be a finite subcomplex of K such that $d_K(K \setminus E_1, E_0) \geq 16$.

Let $\phi : \partial\sigma \to \partial M$, with $diam_M(\phi(\partial\sigma)) \leq b$. Then (use claim 1 and **(1)**)

$$diam_K(c\phi(\partial\sigma)) \leq diam_M(c\phi(\partial\sigma)) + 1 \leq diam_M(\phi(\partial\sigma)) + 2 + 1 \leq b + 3$$

Thus there is a γ such that $c\phi(\partial\sigma) \subset \gamma C_0$. Let g be the map corresponding to γ^{-1} given by claim 2.

Now, for $x \in \partial\sigma$,

$$d_K(cg\phi(x), C_0) \leq d_K(\gamma^{-1}c\phi(x), C_0) + 15 = d_K(c\phi(x), \gamma C_0) + 15 = 15$$

which means that $cg\phi(x) \in C_1$. Thus $g\phi(\partial\sigma) \subset c^{-1}(C_1)$. This implies that there is a map $\phi' : \sigma \to c^{-1}(E_0) \cap \partial M$ extending $g\phi$. Extend now ϕ by defining $\phi = g^{-1}\phi'$. Now, for $x \in \sigma$,

$$d_K(c\phi(x), \gamma E_0) = d_K(cg^{-1}\phi'(x), \gamma E_0) = d_K(\gamma^{-1}cg^{-1}\phi'(x), E_0) \leq d_K(cgg^{-1}\phi'(x), E_0) + 15 = 15$$

consequently $c\phi(x) \in \gamma E_1$. Thus $\phi(\sigma) \subset c^{-1}(\gamma E_1)$, and we get (use **(1)**)

$$diam_M(\phi(\sigma)) \leq diam_M(c^{-1}(\gamma E_1)) \leq diam_M(\gamma E_1) + 2 \leq diam_K(\gamma E_1) + 2 = diam_K(E_1) + 2$$

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and take $a = diam_K(E_1) + 2$, that depends only on b. This completes the proof of the claim.

Claim 4. There is a constant a, such that the following holds. For any pair of finite simplicial complexes $L_0 \subset L$, with $dim(L) \leq (n-1)$, L_0 contains the vertices of L, and a PL map $\psi : L_0 \to \partial M$ with $diam_M(\psi(L_0 \cap \sigma)) \leq 3$, $\sigma \in (L)^1$, there is an extension $\psi : L \to \partial M$, with $diam_M(\psi(\sigma)) \leq a$, for all $\sigma \in L$.

Here $(L)^1$ is the one-skeleton of L. To prove claim 4, proceed as follows. First, extend ψ to $L_0 \cap (L)^1$. For this just choose a geodesic between the endpoints of $\sigma^1 \in (L)^1$. Note that now $diam_K(\psi(\partial \sigma^2)) \leq 6$, for every two-simplex σ^2. Apply claim 3 to $b = 6$, and we get a constant a_6, and an extension of ψ to $L_0 \cap (L)^2$, with $diam_K(\psi(\sigma^2)) \leq a_6$. Then $diam_K(\psi(\partial \sigma^3)) \leq 2a_6$, and apply claim 3 again. We proceed in the same way until we reach dimension $(n-1)$. This proves claim 4.

Now, let $N = a + 2$, where a is the constant from claim 4, and choose a base point $x_0 \in \partial M$.

Recall that we assumed $(M, \partial M)$ to be PL-homeomorphic to $(\mathbb{R}^n_+, \mathbb{R}^{n-1})$. Recall also that (M, d_M) is a proper metric space.

Let A denote the closed ball in M with center x_0 and radius N. Since A is compact, there is a PL $(n-1)$-ball $\tilde{D} \subset \partial M$, such that $A \cap \partial M \subset int \, \tilde{D}$.

Let $\tilde{\rho} : S \to \partial M$, where S is a $(n-2)$-sphere, be a PL-embedding such that $\tilde{\rho}(S) = \partial(\tilde{D})$. Because $x_0 \in int \, \tilde{D}$ we have that $\tilde{\rho} \neq 0 \in \pi_{n-2}(\partial M \setminus \{x_0\})$

Remark also that $\tilde{\rho} = 0 \in \pi_{n-2}(M \setminus A)$. Thus there is a PL extension ρ of $\tilde{\rho}$ with $\rho : D \to M \setminus A$, where D is a closed $(n-1)$-ball and $\partial D = S$. Hence

$$d_M(\rho(v), x_0) \geq N, \quad for \quad all \quad v \in D$$

and

$$\rho|_S \neq 0 \in \pi_{n-2}(\partial M \setminus \{x_0\}).$$

Subdivide D so that $diam_M(\rho(\sigma)) \leq 1$, for any simplex $\sigma \in D$. Denote by $D_0 = \{v_1, ..., v_k\}$ the zero-skeleton of D. Then $v_1, ..., v_k$ are the *vertices* of D. Because of **(2)**, for every v_i we can select a $w_i \in \partial M$ such that $d_M(\rho(v_i), w_i) \leq 1$ (if $\rho(v_i) \in \partial M$ choose $w_i = \rho(v_i)$). We have that if w_i, w_j correspond to the vertices v_i, v_j of a simplex $\sigma \in D$ then

$$d_M(w_i, w_j) \leq d_M(w_i, \rho(v_i)) + d_M(\rho(v_i), \rho(v_j)) + d_M(\rho(v_j), w_j) \leq 3,$$

and since $d_M(\rho(v), x_0) \geq N$, for all $v \in D$, we get

$$d_M(w_i, x_0) \geq d_M(\rho(v_i), x_0) - d_M(\rho(v_i), w_i) \geq N - 1.$$

Define $\psi : S \cup D_0 \to \partial M$, by $\psi(u) = \rho(u)$, for $u \in S$, and $\psi(v_i) = w_i$, for $i = 1, ...k$. Remark that $d_M(\psi(u), x_0) \geq N - 1$, for $u \in S \cup D_0$.

By claim 4 we can extend ψ to a map $\psi : D \to \partial M$, with $diam_M(\psi(\sigma)) \leq a$, for all $\sigma \in D$. Note that, by definition, $\psi|_S = \rho|_S$. Then, for $u \in D$,

$$d_M(\psi(u), x_0) \geq d_M(\psi(v_i), x_0) - d_M(\psi(u), \psi(v_i)) \geq N - 1 - a = (a + 2) - 1 - a = 1 > 0$$

where v_i is a vertex in a simplex that contains u.

This implies that $\psi(D) \subset \partial M \setminus \{x_0\}$. Hence $\psi|_S = 0 \in \pi_{n-2}(\partial M \setminus \{x_0\})$. But this is a contradiction because $\psi|_S = \rho|_S \neq 0 \in \pi_{n-2}(\partial M \setminus \{x_0\})$. This completes the proof of the proposition.

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